THE UNIVERSITY OF ALBERTA

[illegible]

by

[illegible]

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA

[illegible]

THE UNIVERSITY OF ALBERTA

SOME RESULTS ON ARITHMETIC PROGRESSIONS

BY

© ANDREW CHIANG-FUNG LIU

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS

EDMONTON, ALBERTA.

Spring, 1972.

TO MY BROTHER

ABSTRACT

This thesis deals with sets of natural numbers free from arithmetic progressions of a certain length, the starting point being van der Waerden's theorem. We attempt to present a systematic account of established results and related problems, and to compile a comprehensive bibliography.

In Chapter Three, we prove the following new lower bound for the van der Waerden function

$$W(k,t) > k^{c/(\log k)^s}$$

where $s=[\log t/\log 2]$. This improves existing estimates substantially.

In Chapter Four, we study a related problem and prove two new results, which generalize previous work done.

ACKNOWLEDGEMENT

I take this opportunity to thank my supervisor Prof. H. L. Abbott for introducing me to the subject dealt with in this thesis, and for his invaluable help during its preparation. Chapter Three of this thesis will appear in the Journal of Combinatorial Theory as a joint paper of Prof. Abbott and myself, while results in Chapter Four are collaborations between us and Prof. J. Riddell of the University of Victoria.

During our association, Prof. Abbott and Prof. W. G. Brown of McGill University are constant sources of encouragement, and they have helped me overcome many difficulties, academic and otherwise. To both of them, I owe a deep gratitude.

I am indebted to Prof. P. Erdös of the Hungarian Academy of Science for granting me two enlightening discussions, and lending me a manuscript as yet to appear, which contains the most recent developments in the subject.

Helpful information communicated to me by Prof. V. Chvátal of Stanford University and Prof. T. C. Brown of Simon Fraser University are deeply appreciated.

CONTENT

NOTATIONS

The usual mathematical symbols will be used in this thesis. In particular, ~ assumes the dual role of denoting the set-theoretic difference as well as the number-theoretic relation of "being asymptotic to".

As is usual, the letter ε stands for a fixed but arbitrary small positive quantity. The letter c is exclusively reserved for constant, either absolute or depending on some predetermined quantities. Each occurence of ε or c will, and usually does, assume different values.

To facilitate the reading of the text, we adopt the following conventions. By an arithmetic progression is meant an arithmetic progression with distinct terms. The phrase "a subset with n elements" is shortened to "an n-subset".

Other symbols and conventions will be self-evident in the text.

CHAPTER ONE

VAN DER WAERDEN'S THEOREM

§1.

The starting point of our study is a well-known theorem of van der Waerden([37]). "Given natural numbers k and t, there exists a natural number $W(k,t)$ such that if the sequence of natural numbers $1, 2, \ldots, W(k,t)$ is divided in any way into k classes, at least one of the classes contains an arithmetic progression of $t+1$ terms."

This theorem originated from the following conjecture of Baudet. "Arbitrary partition of the sequence of all natural numbers into two classes will enforce the presence of arithmetic progressions of any prescribed length in one of the classes."

Artin, Schreier and van der Waerden([38]) observed that the number of classes of division need not be restricted to two, and that it is not necessary to decompose the entire sequence of natural numbers. Van der Waerden then proved this generalization by an inductive argument on t.

Other proofs were given by Grünwald([22]),

Witt([40]) and Lukomskaja([16]), but they are not essentially different from van der Waerden's proof.

We first make some trivial observations about the van der Waerden function $W(k,t)$. It is clear that for all k and t:

$$W(k,1)=k+1$$

$$W(1,t)=t+1$$

$$W(k,t) \leq W(k+1,t)$$

$$W(k,t) \leq W(k,t+1)$$

Simple experimentation reveals that $W(2,2)=9$.

Ideally we would like to have a general formula for $W(k,t)$, or alternatively some relations which would define $W(k,t)$ recursively. No progress has been made in this direction. In fact, it is extremely doubtful whether such general formula or recurrence relations exist.

It should also be mentioned that the problem of finding upper bounds for $W(k,t)$ is almost hopeless. Theoretically, every proof of van der Waerden's theorem furnishes an upper bound for $W(k,t)$, and conversely every upper bound for $W(k,t)$ found constitutes a proof of van der Waerden's theorem. However, the upper bounds known to date are so large that they are practically

useless. They are not even expressible in terms of the elementary functions. As an illustration, while we know that $W(2,2)=9$, van der Waerden's proof yields a bound which gives $W(2,2)\leq 69$.

There are two main streams of research concerning the function $W(k,t)$.

(i) Determining individual values of $W(k,t)$.

The only non-trivial values of $W(k,t)$ known are $W(2,3)=35$ ([32]) and $W(3,2)=27$ ([7],[8]). The last value prompted Chvátal to conjecture that $W(k,2)=3^k$.

These values are determined via computer facilities. A natural number n such that $n\leq W(k,t)$ is first chosen. All possible partitions of the sequence of natural numbers $1, 2, \ldots, n$ into k classes are examined. If for every partition one class contains an arithmetic progression of $t+1$ terms, then $W(k,t)=n$. Otherwise n is replaced by $n+1$ and the process is repeated.

The initial choice $n=2$ is always valid. Occasionally, mathematical arguments may furnish a non-trivial value. Employing quadratic residues, Folkman ([5]) showed that $W(2,3)\geq 34$. Similar technique can be

applied for other cases, but no general formulation is known.

(ii) Finding lower bounds for $W(k,t)$.

The first non-trivial lower bound is due to Erdös and Rado([12]):

$$W(k,t)>(2tk^{t})^{\frac{1}{2}}$$

They used a simple counting argument. The result is fairly weak. By a difficult and ingenious improvement of their method, Schmidt([33]) obtained:

$$W(k,t)>k^{t-c(t(\log t))^{\frac{1}{2}}}$$

For the case $k=2$, Berlekemp([5]) established, by constructive argument, the following lower bound:

$$W(2,t)>p2^{p} \tag{1.1}$$

where p is the largest prime number less than or equal to t. The proof is based on the Galois field construction. For large t, (1.1) is the best lower bound known.

On the other hand, if k is large and t is small, the above results are all weak compared to:

$$W(k,t)>tk^{c(\log k)} \tag{1.2}$$

This is proved by Moser([20]) using a geometric idea. This bound will be superseded by our result in Chapter Three.

§2.

In this section, we digress from the main discussion to consider some generalizations of van der Waerden's theorem.

Along the same line as for the well-known theorem of Ramsey([23]), we give a more general version of van der Waerden's theorem. "Given natural numbers k, $t_1, t_2, \ldots, t_k$, there exists a natural number $R(k;t_1,t_2,\ldots,t_k)$ such that if the sequence of natural numbers $1, 2, \ldots, R(k;t_1,t_2,\ldots,t_k)$ is divided in any way into k classes, there exists at least one i, $1\leq i\leq k$, such that the i-th class contains an arithmetic progression of t_i+1 terms."

This theorem obviously implies van der Waerden's theorem. It is also implied by van der Waerden's theorem since

$$R(k;t_1,t_2,\ldots,t_k)\leq W(k,t^*)$$

where $t^*=max\{t_1,t_2,\ldots,t_k\}$. Clearly the order of the t's is immaterial. If $t_1=t_2=\ldots=t_k=t$, then

$$R(k;t_1,t_2,\ldots,t_k)=W(k,t)$$

An obvious lower bound is given by

$$R(k;t_1,t_2,\ldots,t_k)\geq W(k,t_*)$$

where $t_*=min\{t_1,t_2,...,t_k\}$. Nothing much is known about this function.

Using computer facilities, Chvátal([7],[8]) found the following numbers $R(2;t_1,t_2)$:

	$t_1 =$ 2	3	4	5	6
$t_2=2$	9	18	22	32	46
3	18	35	55		
4	22	55			
5	32				
6	46				

Another generalization of van der Waerden's theorem, due to Brauer([6]), is stated as follows. "Given natural numbers k and t, there exists a natural number $B(k,t)$ such that if the sequence of natural numbers $1, 2, ..., B(k,t)$ is divided in any way into k classes, at least one of the classes contains an arithmetic progression of $t+1$ terms together with the common difference of the progression."

This statement can easily be deduced from the original theorem by an inductive argument on k. Using this new result, Brauer proved that there exists primes which have blocks of consecutive quadratic residues and

blocks of consecutive quadratic nonresidues of arbitrary lengths.

It is easily seen that for any k and t,

$$B(1,t)=W(1,t)$$

$$B(k,t)\geq W(k,t)$$

$$B(k,t)>t(t+2)^{k-1}$$

The last inequality was improved by Abbott and Hanson ([2]) to

$$B(k,t)>t(2t+1)^{k-1}$$

For $t=2,3,4$, still better bounds were given:

$$B(k,2)>c(253)^{k/3}$$

$$B(k,3)>c(105)^{k/2}$$

$$B(k,4)>c(385)^{k/2}$$

It is interesting to note that if $t=1$, then an arithmetic progression of length $t+1$ together with the common difference of the progression constitutes a set which contains a solution to the equation $x+y=z$, where x and y need not be distinct. Thus Brauer's theorem contains as a special case a well-known theorem of Schur([34]) on sum-free sets. Schur's theorem has been intensively studied by Abbott([1]).

We now consider a result which, though not a direct generalization of van der Waerden's theorem,

contains the theorem as a corollary.

Consider the following equation in r unknowns $x_1, x_2, \ldots, x_r$:

(1.3) $$a_1x_1+a_2x_2+\ldots+a_rx_r=b$$

where the a's are non-zero integers and b is any integer. Rado([22]) called equation (1.3) k-fold regular if there exists a natural number $S(k)$ such that if the sequence of natural numbers $1, 2, \ldots, S(k)$ is divided in any way into k classes, at least one of the classes contains a solution to (1.3). Equation (1.3) is called regular if it is k-fold regular for every natural number k.

Rado's regularity theorem is stated as follows. "Equation (1.3) is regular if and only if some subset of the a's has sum b."

The regularity of the equation $x_1-2x_2+x_3=0$ implies van der Waerden's theorem for the case $t=2$. The regularity theorem can be extended allowing for the replacement of equation (1.3) by a system of linear equations. Then van der Waerden's theorem follows from the regularity of

$$x_i-2x_{i+1}+x_{i+2}=0$$

for $i=1,2,\ldots,t-1$, where t is a given natural number.

The regularity of the equation $x_1+x_2-x_3=0$

implies Schur's theorem. The regularity of other equations has been studied by Abbott([1]) and Salie ([32]).

Finally, it should be mentioned that Graham and Rothschild([15]) proved a profound generalization of Ramsey's theorem([23]) from which van der Waerden's theorem and Schur's theorem can be deduced.

CHAPTER TWO

NONAVERAGING SETS

§1.

A concept strongly related to van der Waerden's theorem is that of a nonaveraging set. A nonaveraging set has been defined as a set of natural numbers containing no three elements in arithmetic progression. We define a t-nonaveraging set to be a set of natural numbers containing no $t+1$ elements in arithmetic progression. In the case $t=2$, we shall retain the original terminology of "nonaveraging set".

Let A be any set of natural numbers and let $A(n)=|\{x\in A|x\leq n\}|$. Define $v_t(n)=max\{A(n)\}$, the maximum taken over all t-nonaveraging sets A. When $t=2$, the subscript in $v_t(n)$ will be omitted.

Erdös and Turán were led to the study of $v_t(n)$ by observing that if one can prove $v_t(n)\leq n/k$, one will obtain the bound $W(k,t)\leq n$. Unfortunately, this objective has not been achieved except for isolated cases.

As is the case of the function $W(k,t)$, no

explicit expression is available for $v_t(n)$. However upper bounds for $v_t(n)$ do exist, and $v_t(n)$ satisfies the triangle inequality

$$v_t(m+n) \leq v_t(m)+v_t(n)$$

This facilitates immensely the determination of individual values of $v_t(n)$. The most complete list up to date is due to Wagstaff([39]), who computed $v(n)$ and $v_3(n)$ for $n\leq 52$, and $v_4(n)$ for $n\leq 31$. Previous efforts are credited to Chvatal([7],[8]), Erdös and Turán([14]), Makowski([18]) and Riddell([27]). Szekeres conjectured that $v(\frac{1}{2}(3^r+1))=2^r$. This has been verified for $r\leq 4$.

The first upper bound for $v(n)$ is due to Erdös and Turán([14]). They used a simple elementary argument to show that

(2.1) $$v(n)<(4/9+\varepsilon)n$$

Refining their argument, they obtained

(2.2) $$v(n)<(3/8+\varepsilon)n$$

Moser([19]) also used a simple elementary argument to show that

(2.3) $$v(n)<2n/5+3$$

which improves (2.1). Using a similar but much longer argument, Moser improved (2.3) to

(2.4) $$v(n)<4n/11+5$$

which is stronger than (2.2). Furthermore, (2.3) and (2.4) have the advantage of being free from ε.

The best upper bound for $v(n)$ is due to Roth([29]), who used analytic methods to obtain

(2.5) $$v(n)<cn/\log\log n$$

The first lower bound for $v(n)$ is due to Salem and Spencer([31])

(2.6) $$v(n)>n^{1-c/\log\log n}$$

Behrend([4]) modified their proof and improved the result to

(2.7) $$v(n)>n^{1-c/(\log n)^{\frac{1}{2}}}$$

While Behrend's argument is non-constructive, Moser ([19]) used an elementary method to construct a nonaveraging set which also satisfies (2.7).

For $t\geq 3$, there are only two known lower bounds for $v_t(n)$. One of them is due to Rankin([24], [26]). Since we shall be quoting his result, we shall give a sketch of the proof in the next section.

The other lower bound is due to Riddell([27], [28]), which comes up as a corollary to a slightly different problem.

Let A be a set of n natural numbers. Among

the t-nonaveraging subsets of A, there is one of maximal cardinality $G_t(A)$. Riddell defined $g_t(n)=min\{G_t(A)\}$, the minimum taken over all sets A of n natural numbers.

He proved that

(2.8) $$g_t(n)>cn^{1-2/(t+1)}$$

For $t=2$, he improved the result to

(2.9) $$g_2(n)>c\sqrt{n}$$

For $t=3$, Erdös and Riddell([11]) proved that

(2.10) $$g_3(n)>cn^{2/3}$$

Erdös conjectured that $g_2(n)>n^{1-\varepsilon}$, and this was proved recently by Szemerédi([11]).

Since $v_t(n)=G_t(\{1,2,...,n\})$, we have

$$v_t(n)\geq g_t(n)$$

Hence (2.8), (2.9) and (2.10) are also lower bounds for $v_t(n)$. The result is especially satisfactory when t is not too small compared with n.

Isolated cases are known where $v_t(n)\neq g_t(n)$. Riddell showed that $v(5)>g_2(5)$([27]) and $v(14)>g_2(14)$ ([11]). It seems unlikely that $v(n)=g_2(n)$ for large n, but this has not been disproved.

In connection with $v_t(n)$, the question arises whether, for a fixed value of t, $\lim_{n\to\infty} v_t(n)/n$ exists or not. This question was answered affirmatively by

Behrend([3]). Let $r(t)=\lim_{n\to\infty} v_t(n)/n$. Clearly $0\le r(t)\le 1$. Behrend showed further that either $r(t)=0$ for all t or $\lim_{t\to\infty} r(t)=1$.

Erdös and Turán conjectured that $r(t)=0$ for all t. The case $t=2$ follows from (2.5). Recently, Szemerédi([36]) gave an ingenious elementary proof of $r(3)=0$, using an indirect argument. Roth([30]) will incorporate Szemerédi's idea into a more analytic proof of $r(3)=0$. The cases $t\ge 4$ remain open.

It is conjectured by Szekeres that $\lim_{n\to\infty} v(n)/n^{1-\varepsilon}=0$, but this is implied false by (2.6) or (2.7). It follows that Szekeres' other conjecture mentioned earlier, that $v(\frac{1}{2}(3^r+1))=2^r$, is also false.

§ 2.

The material in this section is extracted from a paper by Rankin([26]). In this paper, Rankin proved a fairly general result, from which we shall infer a lower bound for the function $v_t(n)$.

We first introduce several definitions. Throughout this section, let a, b, d, r and t be natural numbers and p be a real number such that $d<t$ and $0<p<b$.

An ordered set of t r-dimensional vectors $\vec{x}_i$, $1\le i\le t$, with natural numbers as coordinates, will be called a set of degree d if the d-th differences of the set are all equal and non-zero, ie., if $\sum_{u=1}^{d}(-1)^u\binom{d}{u}\vec{x}_{u+i}$ takes the same non-zero value for $1\le i\le t-d$.

Let A be a set of natural numbers. If no ordered t-subset of A is of degree d, then A is said to be of type (d,t). A is said to be of type $[d,t]$ if A is of type (d^*,t) for every $d^*\le d$.

Denote by $B(a,b,p,r)$ the set of all natural numbers x expressible in the form

$$x=a+x_0+x_1b+\ldots+x_{r-1}b^{r-1} \tag{2.11}$$

where for $0\le j\le r-1$, x_j is a natural number such that $x_j\le p$.

With each such x in $B(a,b,p,r)$ is associated an r-dimensional vector $\vec{x}=\langle x_0,x_1,\ldots,x_{r-1}\rangle$, called the associated vector. Define the norm of $\vec{x}$ to be $|\vec{x}|^2=x_0{}^2+x_1{}^2+\ldots+x_{r-1}{}^2$.

We first prove several lemmas.

<u>Lemma 1</u> Suppose $t>2d+1$, $b>2^d p$ and $\{x_i | 1\le i\le t\}$ is a set of t natural numbers of degree d in $B(a,b,p,r)$. Then the associated vectors $\{\vec{x}_i | 1\le i\le t\}$ form a set of t r-dimensional vectors of degree d.

<u>Proof:</u> The d-th differences of the set $\{x_i\}$ are all equal and non-zero. Hence the $(d+1)$-st differences are all zero. We have for $i=1,2,\ldots,t-d$,

$$\begin{aligned} 0&=\sum_{u=0}^{d+1}(-1)^u\binom{d+1}{u}x_{u+i} \\ &=\sum_{u=0}^{d+1}(-1)^u\binom{d+1}{u}\left(a+\sum_{j=0}^{r-1}x_{u+i,j}b^j\right) \\ &=\sum_{j=0}^{r-1}b^j\sum_{u=0}^{d+1}(-1)^u\binom{d+1}{u}x_{u+i,j} \end{aligned}$$

From this, we have

$$\sum_{j=0}^{r-1}b^j\sum_{\substack{u=0\\ u\ odd}}^{d+1}\binom{d+1}{u}x_{u+i,j} \tag{2.12}$$

$$=\sum_{j=0}^{r-1}b^j\sum_{\substack{u=0\\ u\ even}}^{d+1}\binom{d+1}{u}x_{u+i,j}$$

Since x_{u+i} is in $B(a,b,p,r)$ for $1 \le i \le t-d$, we have $x_{u+i,j} \le p$ for $0 \le j \le r-1$. Hence

$$0 \le \sum_{\substack{u=0 \\ u \text{ odd}}}^{d+1} \binom{d+1}{u} x_{u+i,j} \tag{2.13}$$

$$\le p \sum_{\substack{u=0 \\ u \text{ odd}}}^{d+1} \binom{d+1}{u}$$

$$= 2^d p$$

Similarly, we have

$$0 \le \sum_{\substack{u=0 \\ u \text{ even}}}^{d+1} \binom{d+1}{u} x_{u+i,j} \le 2^d p \tag{2.14}$$

Now $2^d p < b$. By (2.12), (2.13) and (2.14),

$$\sum_{u=0}^{d+1} (-1)^u \binom{d+1}{u} x_{u+i,j} = 0$$

for $0 \le j \le r-1$. Therefore

$$\sum_{u=0}^{d+1} (-1)^u \binom{d+1}{u} \vec{x}_{u+i} = \vec{0}$$

Now the $(d+1)$-st differences of $\{\vec{x}_i\}$ are zero. So the d-th differences are constant. They cannot be all zero as otherwise so would the d-th differences of $\{x_i\}$. Hence $\{\vec{x}_i\}$ is of degree d.□

Lemma 2 Let $\{\vec{x}_i \mid 1 \le i \le t\}$ be a set of t r-dimensional vectors of degree d. Then $\{|\vec{x}_i|^2 \mid 1 \le i \le t\}$ is a set of t natural numbers of degree $2d$.

Proof: Since $\{\vec{x}_i \mid 1 \le i \le t\}$ is of degree d, all d-th

differences of the set are equal and non-zero. By a well-known result in numerical analysis(see [21], for example), we can express $\vec{x}_i$ as a polynomial in i of degree d thus

$$\vec{x}_i = \sum_{u=0}^{d} \vec{a}_u i^u$$

where the $\vec{a}$'s are constant r-dimensional vectors with $\vec{a}_d \neq 0$. Then we can write

$$|\vec{x}_i|^2 = \left| \sum_{u=0}^{d} \vec{a}_u i^u \right|^2$$

This is a polynomial in i of degree $2d$ with leading coefficient $|\vec{a}_d|^2 \neq 0$. Hence the $2d$-th differences of $\{|\vec{x}_i|^2 | 1 \leq i \leq t\}$ are all equal and non-zero. Therefore this set is of degree $2d$.□

Lemma 3 Let s be a natural number such that $t > 2^s d$. Let natural numbers a_q, b_q and r_q, and positive real numbers p_q, $1 \leq q \leq s$, be defined such that $b_q > 2^{d2^{q-1}} p_q$. Let a further natural number a_{s+1} be defined. Let A_{s+1} be the set $\{a_{s+1}\}$. For $q = s, s-1, \ldots, 1$, define A_q to be the maximal subset of $B(a_q, b_q, p_q, r_q)$ such that the associated vectors of its elements have norms in A_{q+1}. Then A_1 is of type $[d,t]$.

Proof: Suppose A_1 is not of type $[d,t]$. Then A_1 has an ordered t-subset $\{x_{1,1}, x_{1,2}, \ldots, x_{1,t}\}$ of degree $d^* \leq d$. These numbers belong to $B(a_1, b_1, p_1, r_1)$. Since

$b_1 > 2^d p_1 > 2^{d^*} p_1$, $\{\vec{x}_{1,1}, \vec{x}_{1,2}, \ldots, \vec{x}_{1,t}\}$ form a set of vectors of degree d^* by Lemma 1.

For $1 \le i \le t$, let $x_{2,i} = |\vec{x}_{1,i}|^2$. Then by Lemma 2, $\{x_{2,1}, x_{2,2}, \ldots, x_{2,t}\}$ is of degree $2d^*$. Furthermore, these numbers lie in A_2, and thus belong to $B(a_2, b_2, p_2, r_2)$. Since we have $b_2 > 2^{d^*2} p_2$, we can apply Lemmas 1 and 2 again. Repeating this procedure, we obtain a set of numbers $\{x_{s+1,1}, x_{s+1,2}, \ldots, x_{s+1,t}\}$ in A_{s+1} of degree $2^s d^*$. But A_{s+1} has only one element, and this is impossible. The contradiction shows that A_1 is of type $[d,t]$.□

Before proving the theorem, we need a further result. Consider the equation in r unknowns

(2.15) $$x_0^2 + x_1^2 + \ldots + x_{r-1}^2 = x$$

where x is a natural number and for $0 \le j \le r-1$, x_j is a natural number such that $x_j \le p$. From the corollary of a theorem of Rankin([25]), we deduce that if $350 \le r \le \frac{1}{3} p^{2/3}$, $10^{12} \le p$ and $|x - \frac{1}{3} p^2 r| \le \frac{1}{3} p^2 r^{\frac{1}{2}}$, then the number of solutions of (2.15) is at least

(2.16) $$\frac{2}{5} p^{r-2} r^{-\frac{1}{2}}$$

We are now ready to state and prove the theorem. The argument is relatively simple, but the calculation,

though routine, is quite involved. Some of the details will be omitted.

<u>Theorem 1</u> Let s be a natural number such that $t>2^s d$. Let n be a natural number and let $N=\{1,2,\dots,n\}$. Then there exists a subset A of N of type $[d,t]$ satisfying

(2.17) $$|A|>n^{1-\tilde{c}/(\log n)^{s/(s+1)}}$$

provided that n is sufficiently large. Here the constant $\tilde{c}$ is given by

(2.18) $$\tilde{c}=(1+\varepsilon)(s+1)2^{s/2}(d(\log 2))^{s/(s+1)}$$

<u>Proof:</u> Let ξ denote $\log n$ and let

(2.19) $$\lambda=2^{s/2}(d(\log 2))^{-1/(s+1)}$$

Define r_q for $1\le q\le s$ by

(2.20) $$r_q=[2^{1-q}\lambda\xi^{1/(s+1)}]$$

Define $p_1, p_2, \dots, p_q$ recursively as follows: $p_1=2^{-d}n^{1/r_1}$ and after $p_1, p_2, \dots, p_q$ have been chosen, define p_{q+1} by

(2.21) $$(2^{d2^q}p_{q+1})^{r_{q+1}}=\tfrac{1}{4}2^{d2^q}p_q{}^2r_q{}^{\frac{1}{2}}$$

Define b_q for $1\le q\le s$ by

(2.22) $$b_q=[2^{d2^{q-1}}p_q]+1$$

Finally, let $a_1=1$ and for $1\le q\le s$ let

(2.23) $$a_{q+1}=[\tfrac{1}{3}p_q{}^2r_q]$$

Let n be chosen sufficiently large so that for $1\le q\le s$

(2.24) $$b_1^{r_1} \le 2n(1-b_1^{-1})$$

(2.25) $$350 \le r_q \le \frac{1}{3} p_q^{r_q - 2}$$

(2.26) $$10^{12} \le p_q$$

We shall define sets A_q, $1 \le q \le s+1$, as in Lemma 3. By (2.22), A_1 is of type $[d,t]$. We need show that A_1 is a subset of N,, and that (2.17) holds for A_1.

Let x be any element of A_1. Then by (2.11) we have

(2.27) $$\begin{aligned} 1 \le x &\le 1 + x_0 + x_1 b_1 + \ldots + x_{r_1 - 1} b_1^{r_1 - 1} \\ &\le 1 + p_1 + p_1 b_1 + \ldots + p_1 b_1^{r_1 - 1} \\ &= 1 + p_1 (b_1^{r_1} - 1)(b_1 - 1)^{-1} \end{aligned}$$

By (2.24) and (2.27), we have $1 \le x \le n$, showing that A_1 is a subset of N.

From (2.19), (2.20), (2.21), (2.22) and (2.23), it can be verified that for $1 \le q \le s$,

(2.28) $$\log b_q \sim \log p_q \sim 2^{q(q+1)/2-1} \lambda^{-q} \xi^{1-q/(s+1)}$$

By (2.21) and (2.28), we have for $1 \le q \le s$,

(2.29) $$b_{q+1}^{r_{q+1}} \le \frac{1}{3} 2^{d2^q} p_q^2 r_q^{\frac{1}{2}} (1 - b_{q+1}^{-1})$$

Let x be an element of A_{q+1}. As before, (2.11) implies

(2.30) $$a_{q+1} \le x \le a_{q+1} + p_{q+1} (b_{q+1}^{r_{q+1}} - 1)(b_{q+1} - 1)^{-1}$$

From (2.29) and (2.30), we have for $1 \le q \le s$,

(2.31) $$\left| x - \frac{1}{3} p_q^2 r_q \right| \le \frac{1}{3} p_q^2 r_q^{\frac{1}{2}}$$

By (2.16), (2.25), (2.26) and (2.31), it follows that for each x in A_{q+1}, there are at least $\frac{2}{5}p_q^{r_q-2} r_q^{-\frac{1}{2}}$ vectors of norm x associated with numbers in A_q. Hence for $1\le q\le s$,

$$|A_q|\ge\frac{2}{5}p_q^{r_q-2} r_q^{-\frac{1}{2}}|A_{q+1}| \tag{2.32}$$

From (2.21) and (2.32), we have

$$|A_1|\ge(\frac{2}{5})^s \prod_{q=1}^{s} p_q^{r_q-2} r_q^{-\frac{1}{2}} \tag{2.33}$$

$$=\frac{2^{2d(2^{s-1}-1)+2} p_1^{r_1}}{(10)^s 2^{d(2r_2+4r_3+\ldots+2^{s-1}r_s)} p_s^2 r_s^{\frac{1}{2}}}$$

$$=c^* ne^{-y}$$

where

$$c^*=4(10)^{-s} 2^{2d(2^{s-1}-1)}$$

and

$$y=2(\log p_s)+\tfrac{1}{2}(\log r_s) + \sum_{q=1}^{s} 2^{q-1} dr_q(\log 2) \tag{2.34}$$

By (2.20), (2.28) and (2.34), we have

$$y \sim \xi^{1/(s+1)}(s+1)d\lambda(\log 2) \tag{2.35}$$
$$= \xi^{1/(s+1)}(s+1)2^{s/2}(d(\log 2))^{s/(s+1)}$$

From (2.18), (2.33) and (2.35), we have

$$|A_1|\ge c^* ne^{-\xi^s(s+1)2^{s/2}(d(\log 2))^{s/(s+1)}}$$

$$=n^{1-\tilde{c}/(\log n)^{s/(s+1)}}$$

which is (2.17), proving the theorem.□

It should be noted that a set of type $[1,t]$ or $(1,t)$ is a $(t-1)$-nonaveraging set. Hence (2.17) yields the following lower bound for $v_t(n)$.

(2.36) $$v_t(n)>n^{1-c/(\log n)^{s/(s+1)}}$$

where $t+1>2^s$. The optimal choice of s is $s=[\log t/\log 2]$.

When $t=2$, (2.36) coincides with (2.7). When $t=3$, the same lower bound holds for $v_3(n)$. For $t\geq 4$, we get a larger lower bound, since we can take $s\geq 2$ so that $s/(s+1)\geq 2/3$.

CHAPTER THREE

A NEW LOWER BOUND FOR THE VAN DER WAERDEN FUNCTION

§1.

In this chapter, we obtain a new lower bound for the van der Waerden function $W(k,t)$. The main tool is the following result which is contained implicitly in a paper of Lorentz([17]) in a less general form, in connection with a different problem in additive number theory.

Lemma 4(Lorentz's Covering Lemma) Let n be a natural number and let $N=\{1,2,\ldots,n\}$. Let A be a subset of N and let $z=|A|$. For any integer λ, define $A_\lambda=\{a+\lambda \mid a\in A\}$ and let $D_\lambda=A_\lambda\cap N$. Then there exists a natural number k satisfying

$$k\le\frac{2n-1}{z}\sum_{u=1}^{z}\frac{1}{u} \tag{3.1}$$

such that N is the union of k of the D's.

Proof: We need only consider λ for which $|\lambda|<n$, as otherwise $D_\lambda=\emptyset$. We shall define the numbers $\lambda(1)$, $\lambda(2)$, $\ldots$, $\lambda(k)$ as follows: $\lambda(1)=0$, and after the numbers $\lambda(1)$, $\lambda(2)$, $\ldots$, $\lambda(r)$ have been defined, select $\lambda(r+1)$

so that $\left|D_{\lambda(r+1)} \sim \bigcup_{i=1}^{r} D_{\lambda(i)}\right|$ is maximal. Let k be the first natural number such that $N=\bigcup_{i=1}^{k} D_{\lambda(i)}$. We shall show that k satisfies (3.1).

Let $B_1=D_{\lambda(1)}$ and for $r\geq 2$, let $B_r=D_{\lambda(r)} \sim \bigcup_{i=1}^{r-1} D_{\lambda(i)}$.

Define the numbers $m(z)$, $m(z-1)$, ..., $m(1)$ as follows: $m(z)$ is the least natural number such that $|B_v|=z$ for $v=1,2,\ldots,m(z)$. After the numbers $m(z)$, $m(z-1)$, ..., $m(u+1)$ have been defined, let $m(u)$ be the largest natural number for which $|B_v|=u$ for $\sum_{i=u+1}^{z} m(i)<v<\sum_{i=u}^{z} m(i)$, provided that such a natural number exists. Otherwise define $m(u)=0$. Clearly

(3.2) $$k=\sum_{u=1}^{z} m(u)$$

Now define a sequence of subsets N_{z+1}, N_z, ..., N_1 of N as follows: $N_{z+1}=N$ and for $1\leq u\leq z$, put $N_u=\{a|a\in N,\ a\notin D_{\lambda(i)}$ for $i=1,2,\ldots,\sum_{j=u+1}^{z} m(j)\}$. Let $f(u)=|N_u|$. Then clearly

(3.3) $$f(1)=0$$

and for $1\leq u\leq z$, $f(u+1)-f(u)=u.m(u)$ or

(3.4) $$m(u)=\frac{f(u+1)-f(u)}{u}$$

From (3.2), (3.3) and (3.4), we have

(3.5)
$$k=\sum_{u=1}^{z} m(u)$$
$$=\sum_{u=1}^{z} \frac{f(u+1)-f(u)}{u}$$
$$=\sum_{u=1}^{z-1} \frac{f(u+1)}{u}\,\frac{1}{u+1} + \frac{f(z+1)}{z}$$

For each λ, $|\lambda|<n$, we have $|D_\lambda \cap N_{u+1}|\le u$ for $1\le u\le z$. Hence

(3.6)
$$\sum_{\lambda=1-n}^{n-1} |D_\lambda \cap N_{u+1}|\le (2n-1)u$$

Since each element of N_{u+1} belongs to exactly z of the D's, we have

(3.7)
$$\sum_{\lambda=1-n}^{n-1} |D_\lambda \cap N_{u+1}|=z.f(u+1)$$

From (3.6) and (3.7), we have for $1\le u\le z$

(3.8)
$$\frac{f(u+1)}{u}\le\frac{2n-1}{z}$$

Now from (3.5) and (3.8), we get
$$k<\frac{2n-1}{z}\left(\sum_{u=1}^{z-1}\frac{1}{u+1} + 1\right)$$
$$=\frac{2n-1}{z}\sum_{u=1}^{z}\frac{1}{u}$$

which is (3.1).□

Lorentz's Covering Lemma has very wide applications. Apart from being featured in Lorentz's original paper, it will form the main line of argument in proving our result in the next section. We shall appeal to it again in Chapter Four. As is evident,

Lorentz's Covering Lemma can be applied to any problem which investigates arithmetic properties that are preserved under translation.

§2.

In one of his papers, Moser([19]) made the following remark. "In van der Waerden's theorem, the $W(k,t)$ is extremely large, and it was thought that a study of $v_t(n)$ would yield better bounds for $W(k,t)$. Unfortunately this hope has not as yet been fulfilled."

With the aid of Lorentz's Covering Lemma, we are ready to take the first step towards this fulfilment. Our result depends heavily on Rankin's result (2.36) on t-nonaveraging sets. If a better bound for $v_t(n)$ appears, it can be substituted in our proof to yield a better bound for $W(k,t)$.

Theorem 2 Let $t \geq 2$ be a natural number and let $s=[\log t/\log 2]$. For any natural number k, we have

$$W(k,t) > k^{c(\log k)^s} \tag{3.9}$$

Proof: Throughout the proof, let t, and consequently s, be fixed. Let n be a natural number and let $N=\{1,2,\ldots,n\}$.

Let A be a t-nonaveraging subset of N. We choose A to be maximal so that $z=|A|=v_t(n)$. Define A_λ and D_λ as in Lorentz's Covering Lemma. Since arithmetic progressions are invariant under translation, each D_λ will still be

a t-nonaveraging set.

Let k be the natural number which is obtained as in Lorentz's Covering Lemma. We have then partitioned N into k t-nonaveraging sets. Hence

$$W(k,t)>n \tag{3.10}$$

Now from (3.1) it follows that

$$k<\frac{cn(\log z)}{z} =\frac{cn(\log v_t(n))}{v_t(n)} \tag{3.11}$$

We observed that $\frac{\log x}{x}$ is a decreasing function for $x>e$. Using this fact together with (2.36) and (3.11), we get

$$\begin{aligned} k&<\frac{cn\left(\log n^{1-c/(\log n)^{s/(s+1)}}\right)}{n^{1-c/(\log n)^{s/(s+1)}}} \\ &<cn^{c/(\log n)^{s/(s+1)}}\log n \\ &<n^{c/(\log n)^{s/(s+1)}} \end{aligned} \tag{3.12}$$

Taking logarithms, we have

$$\begin{aligned} \log k&<\frac{c(\log n)}{(\log n)^{s/(s+1)}} \\ &<c(\log n)^{1/(s+1)} \end{aligned}$$

Raising both sides to the $(s+1)$-th power, we obtain $c(\log k)^{s+1}<\log n$, or

$$\begin{aligned} n&>e^{c(\log k)^{s+1}} \\ &=k^{c(\log k)^{s}} \end{aligned} \tag{3.13}$$

Now let k be a given natural number. Choose n to be the least natural number satisfying (3.1). Then k and n will satisfy (3.10) and (3.13). From this we get (3.9), proving the theorem.$\square$

It is obvious that for $t \geq 4$ and k large compared to t, (3.9) is a substantial improvement over Moser's result (1.2). Even in the cases $t=2,3$, we are able to replace the constant c obtained by Moser by a larger value.

CHAPTER FOUR

RELATED PROBLEMS

§1.

We shall now study a problem first considered by Riddell([27],[28]). Let m, n and t be natural numbers satisfying $m \geq n \geq t \geq 2$. Denote by $h(m,n,t)$ the largest natural number with the property that from every n-subset of $\{1,2,...,m\}$, one can select $h(m,n,t)$ numbers no $t+1$ of which are in arithmetic progression. Clearly $h(n,n,t)=v_t(n)$.

Riddell proved that

$$h([n^a],n,2)>n^{1-c(a/\log n)^{\frac{1}{2}}-c/\log n} \qquad (4.1)$$

where $a \geq 1$ is a real number. He also proved that if $a \geq 3$ and n is sufficiently large, then almost all n-subsets of $\{1,2,...,[n^a]\}$ has a nonaveraging subset A such that

$$|A|>n^{1-c/(\log n)^{\frac{1}{2}}} \qquad (4.2)$$

Riddell's arguments do not generalize to the cases $t \geq 3$. We shall now prove two parallel results extended to the general case.

<u>Theorem 3</u> Let $m \geq n \geq t \geq 2$ be natural numbers and let $s=[\log t/\log 2]$. Then

(4.3) $$h(m,n,t) > nm^{-c/(\log m)^{s/(s+1)}}$$

Proof: Let $N=\{1,2,\ldots,m\}$. Let A be a t-nonaveraging subset of N. We choose A to be maximal, so that $z=|A|=v_t(n)$. Define A_λ and D_λ as in Lorentz's Covering Lemma. Since arithmetic progressions are invariant under translation, each D_λ will still be a t-nonaveraging set. Let k be the natural number which is obtained as in Lorentz's Covering Lemma. We have then partitioned N into k t-nonaveraging sets. Now let S be an n-subset of N. Then for some j, $1\le j\le k$, $|S\cap D_{\lambda(j)}|\ge n/k$. Now $S\cap D_{\lambda(j)}$ is also a t-nonaveraging set. Hence $h(m,n,t)\ge n/k$. From (3.12) we have $k<m^{c/(\log m)^{s/(s+1)}}$. Hence we have $h(m,n,t)>nm^{-c/(\log m)^{s/(s+1)}}$, which is (4.3).□

When $t=2$, $s=1$ and our result gives

$$h([n^a],n,2)>n^{1-c(a/\log n)^{\frac{1}{2}}}$$

This is not essentially different from (4.1), but our constant c is smaller.

Before proving our next theorem, we need introduce the concept of a t-nonaveraging set of intervals. Let t and r be natural numbers. By a t-nonaveraging set of r intervals we mean a set of

intervals $I_j=(u+(x_j-1)v,u+x_jv]$ for $1\leq j\leq r$, where u and v are positive real numbers and $\{x_1,x_2,...,x_r\}$ is a t-nonaveraging set of natural numbers. Let I_{*j} denote $(u+(x_j-1)v,u+(x_j-\frac{1}{2})v]$ and I^*_j denote $(u+(x_j-\frac{1}{2})v,u+x_jv]$.

We now prove several lemmas, which are direct parallels of lemmas in Riddell's paper.

<u>Lemma 5</u> Let t and r be natural numbers and $\{I_j|1\leq j\leq r\}$ be a t-nonaveraging set of intervals. Then the union of the I_*'s contains no arithmetic progressions of $t+1$ terms with terms appearing in more than one of the I_*'s.

<u>Proof:</u> Suppose there is an arithmetic progression $\{a_1,a_2,...,a_{t+1}\}$. If two of its terms lie in the same I_{*j}, then all of the terms must belong to I_{*j} since the common difference of the arithmetic progression is less than the distance between the I_*'s.

Suppose the terms are all in different I_*'s, say a_i in $I_{*j(i)}$ for $1\leq i\leq t+1$. Consider a_i, a_{i+1} and a_{i+2} for $1\leq i\leq t-1$. We have

(4.4) $$u+(x_{j(i)}-1)v<a_i$$

(4.5) $$a_i\leq u+(x_{j(i)}-\tfrac{1}{2})v$$

(4.6) $$u+(x_{j(i+1)}-1)v<a_{i\ 1}$$

(4.7) $$a_{i+1}\leq u+(x_{j(i+1)}-\tfrac{1}{2})v$$

(4.8) $$u+(x_{j(i+2)}-1)v<a_{i+2}$$

(4.9) $$a_{i+2}\leq u+(x_{j(i+2)}-\tfrac{1}{2})v$$

Now $a_i+a_{i+2}=2a_{i+1}$. From (4.4), (4.7) and (4.8), we have

(4.10) $$x_{j(i)}+x_{j(i+2)}<2x_{j(i+1)}+1$$

From (4.5), (4.6) and (4.9), we have

(4.11) $$2x_{j(i+1)}<x_{j(i)}+x_{j(i+2)}+1$$

Now (4.10) and (4.11) imply that $x_{j(i)}+x_{j(i+2)}=2x_{j(i+1)}$. This holds for $1\leq i\leq t-1$. Hence $\{x_{j(1)},x_{j(2)},\ldots,x_{j(t+1)}\}$ forms an arithmetic progression. This contradicts the assumption that $\{I_j|1\leq j\leq r\}$ is a t-nonaveraging set of intervals. Hence the a's cannot appear in more than one of the I_*'s.□

<u>Lemma 6</u> Let t and r be natural numbers and $\{I_j|1\leq j\leq r\}$ be a t-nonaveraging set of intervals. Let A be a set of natural numbers such that $A\cap I_j\neq\emptyset$ for $1\leq j\leq r$. Then A contains a t-nonaveraging subset of cardinality at least $[(r+1)/2]$.

<u>Proof:</u> By Lemma 5, the union of the I_*'s contains no arithmetic progressions of $t+1$ terms with terms appearing in more than one of the I_*'s. A similar statement about the union of the I^*'s can be proved.

Since $A\cap I_j\neq\emptyset$ for $1\leq j\leq r$, we can pick one number from each $A\cap I_j$. The either the union of the I_*'s or the union of

the I^*'s must contain at least $[(r+1)/2]$ of these numbers. This then is a t-nonaveraging set.□

<u>Lemma 7</u> Let m and d be natural numbers such that $m\geq d$. Let $N=\{1,2,\ldots,m\}$. Let $w=m/d$ and partition N into the intervals

$$(0,w],(w,2w],\ldots,((d-1)w,dw] \tag{4.12}$$

Let n and p be natural numbers and let $b(p,n)$ denote the number of n-subsets of N that have elements appearing in fewer than p of the intervals in (4.12). Then

$$b(p,n)<(n!)^{-1}2^n w^n d^p p^{n+1} \tag{4.13}$$

<u>Proof:</u> Denote by $f(j)$ the number of n-subsets of N that have elements appearing in exactly j of the intervals in (4.12). Then

$$f(j)\leq \binom{d}{j}\sum \binom{[w+1]}{b_1}\binom{[w+1]}{b_2}\cdots\binom{[w+1]}{b_j} \tag{4.14}$$

the summation taken over all possible partitions of n into j natural numbers $n=b_1+b_2+\ldots+b_j$.

Observing that $a^b>\frac{a^b}{b!}>\binom{a}{b}$, (4.14) gives

$$f(j)\leq d^j\sum\frac{(w+1)^n}{b_1!b_2!\ldots b_j!} \tag{4.15}$$

By the multinomial theorem, (4.15) becomes

$$\begin{aligned} f(j) &\leq (n!)^{-1}(w+1)^n d^j j^n \\ &\leq (n!)^{-1}2^n w^n d^j j^n \end{aligned} \tag{4.16}$$

Now $b(p,n)=\sum_{j=1}^{p-1} f(j)$. Hence from (4.16), we have

$$b(p,n)<(n!)^{-1}2^n w^n \sum_{j=1}^{p-1} d^j j^n$$

$$<(n!)^{-1}2^n w^n d^p p^{n+1}$$

which is (4.13), proving the lemma.□

<u>Lemma 8</u> Let n be a natural number and let $d=[n^{1+\varepsilon}]$. Let m be a natural number such that $m>d$, and let $w=m/d$. Let $N=\{1,2,...,m\}$ and let it be partitioned into the intervals in (4.12). Then almost all n-subsets of N have elements in common with at least $[\varepsilon n/(1+\varepsilon)]$ of the intervals in (4.12).

<u>Proof:</u> The total number of n-subsets of N is given by

$$\binom{m}{n}=\frac{m(m-1)...(m-n+1)}{n!}$$

$$>\frac{(m-n)^n}{n!}$$

$$>\frac{m^n}{n!}(1-\frac{1}{n^\varepsilon})^n$$

$$>\frac{m^n}{n!}\left[(1-\frac{1}{n^\varepsilon})^{n^\varepsilon}\right]^{n^{1-\varepsilon}}$$

$$>\frac{m^n}{n!}(\frac{1}{e})^n$$

On the other hand, the total number of n-subsets of N having elements in common with fewer than $[\varepsilon n/(1+\varepsilon)]$ of the intervals in (4.12) is given by $b([\varepsilon n/(1+\varepsilon)],n)$. By Lemma 7, we have, with p denoting $[\varepsilon n/(1+\varepsilon)]$,

$$b([\varepsilon n/(1+\varepsilon)],n)=(n!)^{-1}2^n w^n d^p p^{n+1}$$

$$<\frac{2^n m^n n^{\varepsilon n}\varepsilon^{n+1} n^{n+1}}{(n!)n^{(1+\varepsilon)n}(1+\varepsilon)^{n+1}}$$

$$<\frac{m^n}{n!}\ \frac{n}{2}\left(\frac{2\varepsilon}{1+\varepsilon}\right)^{n+1}$$

To show that $b([\varepsilon n/(1+\varepsilon)],n)=o\left(\binom{m}{n}\right)$, we need show that $\frac{n}{2}\left(\frac{2\varepsilon}{1+\varepsilon}\right)^{n+1}=o\left(\left(\frac{1}{e}\right)^n\right)$, or $\frac{n\varepsilon}{1+\varepsilon}\left(\frac{2e\varepsilon}{1+\varepsilon}\right)^n=o(1)$. This can be done by picking $\varepsilon<\frac{1}{2e-1}$, so that $\frac{2e\varepsilon}{1+\varepsilon}<1$.□

<u>Theorem 4</u> Let m and n be natural numbers such that $m>n^{1+\varepsilon}$. Let $N=\{1,2,...,m\}$. Let t be a given natural number and let $s=[\log t/\log 2]$. Then almost all n-subsets of N contain a t-nonaveraging subset A such that

(4.17) $$|A|>n^{1-c/(\log n)^{s/(s+1)}}$$

<u>Proof:</u> Let $d=[n^{1+\varepsilon}]$ and $w=m/d$. By Lemma 8, almost all n-subsets of N have elements in common with at least $[\varepsilon n/(1+\varepsilon)]$ of the intervals in (4.12). Let S be such a subset. At least $h([n^{1+\varepsilon}],[\varepsilon n/(1+\varepsilon)],t)$ of these $[\varepsilon n/(1+\varepsilon)]$ intervals form a t-nonaveraging set of intervals. By Lemma 6, S contains a t-nonaveraging subset A such that

(4.18) $$\begin{aligned}|A|&\geq[(h([n^{1+\varepsilon}],[\varepsilon n/(1+\varepsilon)],t)+1)/2]\\&>\tfrac{1}{2}h([n^{1+\varepsilon}],[\varepsilon n/(1+\varepsilon)],t)\end{aligned}$$

By Theorem 3, (4.18) yields

$$\begin{aligned}|A|&>\frac{\varepsilon n}{1+\varepsilon}(n^{1+\varepsilon})^{-c/(\log n^{1+\varepsilon})^{s/(s+1)}}\\&>n^{1-c/(\log n)^{s/(s+1)}}\end{aligned}$$

which is (4.17), proving the theorem.□

If we let $t=2$, then (4.17) becomes (4.2), although again our constant c is larger.

§2.

To conclude our study, we review a number of other problems related to van der Waerden's theorem. We shall not be able to consider all of them. Most problems of this type have been reviewed by Erdös([9],[11]).

Problem (i) Modified van der Waerden function.

Let t be a natural number and a be a real number such that $\frac{1}{2}\leq a\leq 1$. Denote by $f_t(a)$ the least natural number such that if the sequence of natural numbers $1, 2, \ldots, f_t(a)$ are divided in any way into two classes, there exists an arithmetic progression of $t+1$ terms with $a(t+1)$ of them belonging to the same class. Clearly $f_t(1)=W(2,t)$. Erdös([10]) proved, by probability methods, that

$$f_t(a)>(1+\varepsilon)^{t+1}$$

He conjectured that $f_t(a)<a^{t+1}$ if a is sufficiently closed to $\frac{1}{2}$.

Problem (ii) Sets free from arithmetic means.

Straus([35]) defined a nonaveraging set A as a set of natural numbers no one of which is the arithmetic mean of any subset of A with more than one element. To

distinguish this meaning from our definition, we call such a set S-nonaveraging.

Let $f(n)$ denote the maximal number of natural numbers in a S-nonaveraging subset of $\{1,2,...,n\}$ where n is any given natural number. Clearly $v(n) \geq f(n)$. Straus proved that

$$f(n) > n^{c/(\log n)^{\frac{1}{2}}}$$

On the other hand, Erdös and Straus([13]) showed that

$$f(n) < cn^{3/4}$$

Using a recent result of Szemerédi, they improved this to

$$f(n) < cn^{2/3}$$

Straus conjectured that $f(n) < n^{c/(\log n)^{\frac{1}{2}}}$.

Problem (iii) Lattice points on a straight line.

Let n and t be natural numbers. Denote by $f_t(n)$ the largest natural number such that one can find $f_t(n)$ lattice points in the n-dimensional cube $\{<x_1,x_2,...,x_n> | 0 \leq x_i \leq t \text{ for } 1 \leq i \leq n\}$, no $t+1$ of which are on a straight line. Clearly $v_t((t+1)^n) \leq f_t(n)$. Moser([11]) showed that

$$f_2(n) > c3^n/\sqrt{n}$$

Riddell([27]) proved that

$$f_t(n) > c\sqrt{t}(t+1)^{n+1}/\sqrt{n}$$

The two results coincide when $t=2$.

BIBLIOGRAPHY

[1] H. L. Abbott

Some problems on combinatorial analysis

Ph. D. Thesis, Univ. Alberta(1965)

[2] H. L. Abbott and D. Hanson

Lower bounds for certain types of van der Waerden numbers

J. Comb. Theory All (1972) 143-146

[3] F. A. Behrend

On sequences of integers containing no arithmetic progressions

Casopis Mat. Fys. 67 (1938) 235-238

[4] F. A. Behrend

On sets of integers which contain no three terms in arithmetic progression

Proc. Nat. Acad. Sci. U. S. A. 32 (1946) 331-332

[5] E. R. Berlekemp

A construction for partitions which avoid long arithmetic progressions

Canad. Math. Bull. 11 (1968) 409-414

[6] A. Brauer

Uber Sequenzen von Potenzresten(I)

Sitzungsber. Preuss. Akad. Wiss. Phys. Math. 16 (1928) 9-16

Uber Sequenzen von Potenzresten(II)

Sitzungsber. Preuss. Akad. Wiss. Phys. Math. 19 (1931) 329-341

[7] V. Chvátal

Some unknown van der Waerden numbers

"Combinatorial Structures and their Applications" ed. Guy et al., Gordon & Breach(1969) 31-33

[8] V. Chvátal

Hypergraphs and Ramseyian theorems

Ph. D. Thesis, Univ. Waterloo(1971)

[9] P. Erdös

Some unsolved problems

Michigan Math. J. 4 (1957) 291-300

Some unsolved problems

Publ. Math. Inst. Hung. Acad. Sci. 6 (1961) 221-254

Some recent advances and current problems in number theory

"Lectures on Modern Mathematics" vol. 3, ed. Saaty, Wiley(1965) 196-244

Extremal problems in number theory

Proc. Symp. Pure Math. A. M. S. 8 (1965) 181-189

[10] P. Erdös

On combinatorial problems concerning a theorem of Ramsey and van der Waerden(in Hungarian)

Mat. Lapok 14 (1963) 29-37

[11] P. Erdös

Problems and results on combinatorial number theory

To appear.

[12] P. Erdös and R. Rado

Combinatorial theorems on classifications of subsets of a given set

Proc. London Math. Soc. (3) 2 (1952) 417-439

[13] P. Erdös and E. G. Straus

Nonaveraging sets(II)

"Combinatorial Theory and its Applications" vol. 2, ed. Erdös et al., Colloq. Math. Soc. J. Bolyai, 4, North-Holland(1970) 405-411

[14] P. Erdös and P. Turán

On some sequences of integers

J. London Math. Soc. 11 (1936) 261-264

[15] R. L. Graham and B. L. Rothschild

Ramsey's theorem for n-dimensional arrays

Bull. Amer. Math. Soc. 75 (1969) 418-422

Ramsey's theorem for n-parameter sets: an outline

"Combinatorial Theory and its Applications" vol. 2,

ed. Erdös et al., Colloq. Math. Soc. J. Bolyai, 4, North-Holland(1970) 531-552

[16] A. Y. Khinchin
Van der Waerden's theorem on arithmetic progressions
"Three Pearls of Number Theory" by Khinchin, Graylock(1952) 11-17

[17] G. G. Lorentz
On a problem of additive number theory
Proc. Amer. Math. Soc. 5 (1954) 838-841

[18] A. Makowski
Remark on a paper of Erdös and Turán
J. London Math. Soc. 34 (1959) 480

[19] L. Moser
On nonaveraging sets of integers
Canad. J. Math. 5 (1953) 245-252

[20] L. Moser
On a theorem of van der Waerden
Canad. Math. Bull. 3 (1960) 23-25

[21] K. L. Nielson
Differences of polynomial
"Methods in Numerical Analysis" by Nielson, MacMillan(1964) 2nd ed. 44-46

[22] R. Rado
Verallgemeinerung eines Satzes von van der Waerden

Sitzungsber. Preuss. Akad. Wiss. Phys. Math. 21 (1933) 589

Studien zur Kombinatorik

Math. Z. 36 (1933) 424-480

Note on combinatorial analysis

Proc. London Math. Soc. (2) 48 (1939) 122-160

Some partition theorems

"Combinatorial Theory and its Applications" vol. 3, ed. Erdös et al., Colloq. Math. Soc. J. Bolyai, 4, North-Holland(1970) 929-936

[23] F. P. Ramsey

On a problem of formal logic

Proc. London Math. Soc. (2) 30 (1928) 264-286

[24] R. A. Rankin

On sequences of integers containing no arithmetic progressions

Bull. Malay Math. Soc. 8 (1961) 43-52

[25] R. A. Rankin

Representations of a number as the sum of a large number of squares

Proc. Roy. Soc. Edinburgh A65 (1962) 318-331

[26] R. A. Rankin

Sets of integers containing not more than a given number of terms in arithmetic progression

Proc. Roy. Soc. Edinburgh A65 (1962) 332-344

[27] J. Riddell

Problems in combinatorial and additive number theory

Ph. D. Thesis, Univ. Alberta(1967)

[28] J. Riddell

On sets of numbers containing no l terms in arithmetic progression

Nieuw Arch. Wisk. (3) 17 (1969) 204-209

[29] K. F. Roth

Sur quelques ensembles d'entiers

C. R. Acad. Sci. Paris 234 (1952) 388-390

On certain sets of integers(I)

J. London Math. Soc. 28 (1953) 104-109

On certain sets of integers(II)

J. London Math. Soc. 29 (1954) 20-26

[30] K. F. Roth

Remark concerning integer sequences

Acta Arith. 9 (1964) 257-260

Irregularities of sequences relative to arithmetic progressions(I)

Math. Ann. 169 (1967) 1-25

Irregularities of sequences relative to arithmetic progressions(II)

Math. Ann. 174 (1967) 41-52

Irregularities of sequences relative to arithmetic progressions(III)

J. Number Theory 2 (1970) 125-142

[31] R. Salem and D. C. Spencer

On sets of integers which contain no three terms in arithmetic progression

Proc. Nat. Acad. Sci. U. S. A. 28 (1942) 561-563

On sets which do not contain a given number of terms in arithmetic progression

Nieuw Arch. Wisk. (2) 23 (1950) 133-143

[32] H. Salie

Zur Verteilung naturlicher Zahlen auf elementfremde Klassen

Sitzungsber. Verh. Sachs. Akad. Wiss. Leipzig 4 (1954) 2-26

[33] W. M. Schmidt

Two combinatorial theorems on arithmetic progressions

Duke Math. J. 29 (1962) 129-140

[34] I. Schur

Uber die Kongruenz $x^m+y^m \equiv z^m \pmod p$

Jber. Deutsch. Math. Verein. 25 (1917) 114-117

[35] E. G. Straus

Nonaveraging sets(I)

Proc. Symp. Pure Math. A. M. S. 19 (1971) 215-222

[36] E. Szemerédi

On sets of integers containing no four elements in arithmetic progression

Acta Math. Acad. Sci. Hungar. 20 (1969) 89-104

On sets of integers containing no four elements in arithmetic progression

"Number Theory", ed. Turán, Colloq. Math. Soc. J. Bolyai, 2, North-Holland(1970) 197-204

[37] B. L. van der Waerden

Beweis einer Baudetschen Vermutung

Nieuw Arch. Wisk. 15 (1927) 212-216

[38] B. L. van der Waerden

Wie der Beweis der Vermutung von Baudet gefunden wurde

Abh. Math. Sem. Univ. Hamburg 28 (1965) 6-15

How the proof of Baudet's conjecture was found

"Studies in Pure Mathematics", ed. Mirsky, Academic Press(1971) 251-260

[39] S. S. Wagstaff, Jr.

On sequences of integers with no four or no five numbers in arithmetic progression

Math. Comp. 21 (1967) 695-699

[40] E. Witt

Ein kombinatorischer Satz der Elementargeometrie

Math. Nachr. 6 (1952) 261-262